A HARMONIC ANALYSIS APPROACH TO ESSENTIAL NORMALITY OF PRINCIPAL SUBMODULES

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Abstract. Guo and the second author have shown that the closure $[I]$ in the Drury-Arveson space of a homogeneous principal ideal I in $\mathbb{C}[z_1, \cdots, z_n]$ is essentially normal. In this note, the authors extend this result to the closure of any principal polynomial ideal in the Bergman space. In particular, the commutators and cross-commutators of the restrictions of the multiplication operators are shown to be in the Schatten p-class for $p > n$. The same is true for modules generated by polynomials with vector-valued coefficients. Further, the maximal ideal space X_I of the resulting C^*-algebra for the quotient module is shown to be contained in $Z(I) \cap \partial\mathbb{B}_n$, where $Z(I)$ is the zero variety for I, and to contain all points in $\partial\mathbb{B}_n$ that are limit points of $Z(I) \cap \mathbb{B}_n$. Finally, the techniques introduced enable one to study a certain class of weight Bergman spaces on the ball.

1. Introduction

In [3, 4] Arveson raised the interesting question of whether homogeneous polynomial ideals lead to C^*-algebras of essentially normal operators. In particular, one knew that for Hilbert spaces of holomorphic functions on the open unit ball $\mathbb{B}_n = \{z \in \mathbb{C}^n : |z| < 1\}$ such as the Hardy and Bergman spaces, the operators defined to be multiplication by polynomials were essentially normal. Arveson focused on a related space, now called the Drury-Arveson space, H_n^2, and showed the same was true. Moreover, he asked if the submodule $[I]$ defined as the closure of a homogeneous polynomial ideal I has the same property. Actually, the commutators and cross-commutators of these multiplication operators on H_n^2 are in the Schatten $p-$class $\mathcal{L}^p$ for $p > n$ and Arveson asked if the same was true for the operators on $[I]$. Perhaps the best result responding to this question is due to Guo and the second author [17], which established that Arveson's conjecture is valid for principal *homogeneous* polynomial ideals. In this paper, we introduce a new approach to this problem based on covering techniques from harmonic analysis. We use it to extend the earlier result to arbitrary principal polynomial ideals.

Theorem. *If $\mathcal{M} = [p]$ is the submodule of the Bergman space $L_a^2(\mathbb{B}_n)$ generated by an analytic polynomial p, then $\mathcal{M}$ is $p-$essentially normal for $p > n$.*

As in [17], we show that the $p-$essential normality extends to submodules generated by a polynomial with vector-valued coefficients.

Although the overall strategy in this paper is similar to that used in [17], the techniques used in this paper are very different and, we believe, provide better insight into

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why the result is true. In particular, the key step in the proof in [17] is an inequality which allows one to show that the commutators in question are in the Schatten $p-$class $\mathcal{L}^p$. We refer the reader to the discussions in [13, 20]. An attempted proof of this inequality, using standard techniques from PDE, fails since the estimate obtained only shows that those operators are bounded. Hence a different approach was used in [17], but one which was far from transparent.

Here we take advantage of the fact that the analysis takes place not just in the context of real analytic functions but for holomorphic ones. Hence, we are able to replace the inequality by one involving both the radial and complex tangential derivatives and then modify and extend known techniques from harmonic analysis to obtain the desired result. The key step in this proof rests on weighted norm estimations, which follow from a covering argument, now standard in harmonic analysis, due to Grellier [15]. This approach provides a new proof for the case of principal homogeneous polynomial ideals. However, for general polynomials, there is still a critical step needed. To handle this case, one must replace the quantity estimated in the basic inequality by an infinite series of terms, each one of which requires an estimate involving an analogue of an inequality that follows from this covering argument. To show that the series converges, one needs to examine carefully how the constants in the estimates behave and show that they depend only on the dimension of the ball and the degree of the polynomial.

As a consequence of the essential normality of the cyclic submodule generated by a polynomial, one obtains an extension of the C^*-algebra of compact operators by the algebra of continuous functions on a closed subset of the unit sphere in $\mathbb{C}^n$ which is related to the zero variety of the polynomial. (Here one is using the quotient module defined by $[p]$.) As a result one obtains an odd K-homology element. We discuss these issues as well as other consequences of the main result. In particular, the main result is equivalent to the fact that for the Bergman space defined relating to the volume measure weighted by the square of the absolute value of the polynomial, the commutators of the multiplication operators by coordinate functions on this closure are in $\mathcal{L}^p$ for $p > n$. The result involves an explicit characterization of the elements in the spaces.

In Section 2 we provide the variant inequality, state the norm estimates required and outline the argument of the main result. The norm estimates are established by an appropriate covering argument in Section 3. Finally, in Section 4 we discuss briefly the result for the weighted Bergman space and some of the consequences of essential normality including the odd K-homology element defined.

2. Main Result

In this paper, we are mainly concerned with the (weighted) Bergman spaces $L^2_a(\mathbb{B}_n)$ ($L^2_{a,t}(\mathbb{B}_n)$) over the unit ball $\mathbb{B}_n$. The weighted Bergman space $L^2_{a,t}(\mathbb{B}_n)$ ($t \geq 0$) consists of the analytic functions in $L^2_t(\mathbb{B}_n)$ with the norm

$$\|f\|_t^2 = \int_{\mathbb{B}_n} |f(z)|^2 c_t (1-|z|^2)^t dv(z),$$

where $c_t = \frac{(n+t)!}{n!t!}, dv(z) = \frac{dm(z)}{Vol(\mathbb{B}_n)}$ and dm is the Lebesgue measure over $\mathbb{B}_n$, $Vol(\cdot)$ is the measure of the domain. (In this paper we need only the case that t is a non-negative integer.) It's well known that $L^2_{a,t}(\mathbb{B}_n)$ has the canonical orthogonal basis $\{z^\alpha : \alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{Z}^n, \alpha_i \geq 0 \ for \ 1 \leq i \leq n\}$ (see e.g. [21]) with

$$\|z^\alpha\|_t^2 = \frac{\alpha!(n+t)!}{(n+t+|\alpha|)!},$$

where $\alpha! = \alpha_1! \cdots \alpha_n!$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$ for a multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$.

We will focus on the operators on $L^2_{a,t}(\mathbb{B}_n)$ rather than the function theory. We pursue the same basic strategy as in [17]. For $f \in H^\infty(\mathbb{B}_n)$, the set of all bounded analytic functions on $\mathbb{B}_n$, define the multiplication operator $M_f^{(t)}$ on $L^2_{a,t}(\mathbb{B}_n)$ as

$$M_f^{(t)}(g) = fg, g \in L^2_{a,t},$$

which is a bounded operator with norm $\|f\|_\infty$. And define the weighted Toeplitz operator $T_f^{(t)}$ on $L^2_{a,t}(\mathbb{B}_n)$ with the symbol $f \in L^\infty(\mathbb{B}_n)$ as

$$T_f^{(t)}(g) = P^{(t)} M_f^{(t)}(g) = P^{(t)}(fg), g \in L^2_{a,t},$$

where $P^{(t)}$ is the orthogonal projection from $L^2_t(\mathbb{B}_n)$ to $L^2_{a,t}(\mathbb{B}_n)$. To simplify the notation, we let $\|f\|, M_f, T_f$ denote the norm of f, the multiplication operator and the Toeplitz operator on $L^2_a(\mathbb{B}_n)$, respectively.

In this section we will prove that the cyclic submodule $\mathcal{M} = [p]$, which is generated by an analytic polynomial p in the Bergman space $L^2_a(\mathbb{B}_n)$, is essentially normal ($p-$essentially normal). That is, the commutators $[S_{z_i}, S^*_{z_j}]$ are compact (in $\mathcal{L}^p$) for $1 \leq i, j \leq n$, where S_{z_i} is the restriction of M_{z_i} to $\mathcal{M}$.

In what follows denote by N the number operator on $L^2_a(\mathbb{B}_n)$ as in [2, 17] so that $N(z^\alpha) = |\alpha| z^\alpha$ for any non-negative multi-index α, and let $\partial_i = \partial_{z_i}, \bar{\partial}_i = \partial_{\bar{z}_i}$ be the partial derivatives with respect to $z_i, \overline{z_i}$, respectively. Furthermore, let $R(f) = \sum_{i=1}^n z_i \partial_i(f)$ be the radial derivative. Obviously, $Rf = mf$ for any homogeneous analytic polynomial f with $m = \deg(f)$. We refer the reader to [21] for more properties of the radial derivative. Finally, let $L_{j,i}p = \bar{z}_i \partial_j p - \bar{z}_j \partial_i p$ be the complex tangential derivative, which behaves well relative to the distance to the boundary as shown, for example, in [15], [21, Section 7.6] as well as in other references.

Our first result is a variant of formula (2.6) in [17], which is an identity relating the commutator of multiplication operators and the radial derivative.

Proposition 2.1. *For analytic polynomials $f, p \in \mathbb{C}[z_1, \cdots, z_n]$, the equation*

$$M^*_{z_j} M_p f - M_p M^*_{z_j} f = \sum_{k=0}^\infty \frac{1}{(N+1+n)^{k+1}} [(M_{\partial_j R^k p} - M^*_{z_j} M_{R^{k+1} p}) f], \ 1 \leq j \leq n$$

holds on the Bergman space $L^2_a(\mathbb{B}_n)$.

Proof. By linearity, it is enough to verify the case in which $p = z^\alpha$ and $f = z^\beta$. Using the fact that $M^*_{z_j}(z^\alpha) = \frac{\alpha_j}{n+|\alpha|} z^{\alpha - \varepsilon_j}$, where $1 \leq j \leq n$ and ε_j is the multi-index with a

1 in the j position and 0 in all other positions, then we have

$$\begin{aligned} LHS &= M_{z_j}^* z^{\alpha+\beta} - z^\alpha M_{z_j}^* z^\beta = [\frac{\alpha_j+\beta_j}{n+|\alpha|+|\beta|} - \frac{\beta_j}{n+|\beta|}] z^{\alpha+\beta-\varepsilon_j} \\ &= \frac{\alpha_j(n+|\beta|)-\beta_j|\alpha|}{(n+|\alpha|+|\beta|)(n+|\beta|)} z^{\alpha+\beta-\varepsilon_j}. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} RHS &= \sum_{k=0}^{\infty} \frac{1}{(N+1+n)^{k+1}} \left[|\alpha|^k M_{\partial_j(z^\alpha)} z^\beta - M_{z_j}^*(|\alpha|^{k+1} z^{\alpha+\beta})\right] \\ &= \sum_{k=0}^{\infty} \frac{|\alpha|^k}{(|\alpha|+|\beta|+n)^{k+1}} [\alpha_j - \frac{|\alpha|(\alpha_j+\beta_j)}{n+|\alpha|+|\beta|}] z^{\alpha+\beta-\varepsilon_j} \\ &= \frac{1}{n+|\beta|} [\alpha_j - \frac{|\alpha|(\alpha_j+\beta_j)}{n+|\alpha|+|\beta|}] z^{\alpha+\beta-\varepsilon_j} = LHS, \end{aligned}$$

which completes the proof. □

To use the strategy of [17], we need to show the convergence of the infinite sum in the RHS above in an appropriate sense. The following proposition will play an important role in that.

Proposition 2.2. *For positive integers n and m, there is a positive constant $C(n,m) > 1$ such that for every analytic polynomial $p \in \mathbb{C}[z_1, \cdots, z_n]$ with degree m, the following inequalities hold:*

(1) $\|(R^l p)\, f\|_{2k}^2 \le \dfrac{c_{2k} C(n,m)^{k+1}}{c_{2k-2l}} \|pf\|_{2k-2l}^2$, *for every integer l with $0 \le l \le k$,*

(2) $\|(L_{j,i} p)\, f\|_{2k+1}^2 \le \dfrac{c_{2k+1} C(n,m)^{k+1}}{c_{2k}} \|pf\|_{2k}^2$, *for integers i, j with $1 \le i \ne j \le n$,*

(3) $\|(\partial_j p)\, f\|_{2k+2}^2 \le \dfrac{c_{2k+2} C(n,m)^{k+1}}{c_{2k}} \|pf\|_{2k}^2$, *for every integer j with $1 \le j \le n$,*

for any analytic polynomial $f \in \mathbb{C}[z_1, \cdots, z_n]$ and non-negative integer k, where $c_t = \frac{(n+t)!}{n!t!}$ for $t \in \mathbb{N}$.

The proof of this proposition rests heavily on techniques from harmonic analysis. We postpone the proof to the next section. We show first how to obtain the essential normality of $\mathcal{M} = [p]$ from it.

Lemma 2.3. *Fix $l \in \mathbb{N}$. For any analytic polynomial f satisfying $\partial_\alpha f(0) = 0$ for $|\alpha| < l$ and any non-negative integer k, we have*

(1) $\|\dfrac{1}{(N+1+n)^{k+1/2}} f\|^2 \le \dfrac{(n+2k+1+l)^l}{(l+1+n)^{2k+1}} \|f\|_{2k+1}^2$, *and*

(2) $\|\dfrac{1}{(N+1+n)^{k+1/2}} [T_{z_j}^* - T_{z_j}^{(2k+1)*}](f)\|^2 \le \dfrac{(n+2k+2+l)^l}{(l+n)^{2k+1}} \|f\|_{2k+2}^2, 1 \le j \le n.$

Proof. By the orthogonality of homogeneous polynomials of different degrees, it's enough to show the inequality in the case that f is an analytic homogeneous polynomial with $d = \deg(f) \geq l$.

(1) By the fact that $\|z^\alpha\|_t^2 = \frac{\alpha!(n+t)!}{(n+|\alpha|+t)!}$, we have for a homogeneous analytic polynomial $f = \sum_{|\alpha|=d} a_\alpha z^\alpha$ and a non-negative integer t, that

$$\begin{aligned} \|f\|^2 = \sum_{|\alpha|=d} |a_\alpha|^2 \frac{\alpha! n!}{(n+d)!} &= \frac{n!}{(n+d)!} \frac{(n+t+d)!}{(n+t)!} \sum_{|\alpha|=d} |a_\alpha|^2 \frac{\alpha!(n+t)!}{(n+d+t)!} \\ &= \frac{n!}{(n+d)!} \frac{(n+t+d)!}{(n+t)!} \|f\|_t^2. \end{aligned}$$

Therefore, for a non-negative integer k we have

$$\begin{aligned} LHS^{(1)} &= \|\frac{1}{(d+1+n)^{k+1/2}} f\|^2 = \frac{1}{(d+1+n)^{2k+1}} \|f\|^2 \\ &= \frac{1}{(d+1+n)^{2k+1}} \frac{n!}{(n+d)!} \frac{(n+2k+1+d)!}{(n+2k+1)!} \|f\|_{2k+1}^2. \end{aligned}$$

Here, $LHS^{(1)}$ refers to the left-hand side of the inequality in statement (1).

Since $d \geq l$ and

$$\frac{(n+2k+1+d)!}{(d+1+n)^{2k+1}(n+d)!} = \frac{(n+2k+1+d)\cdots(n+d+1)}{(d+1+n)^{2k+1}} = (1 + \frac{2k}{d+1+n})\cdots(1),$$

we see that this product is monotonically decreasing with respect to d. Thus we have

$$\frac{(n+2k+1+d)!}{(d+1+n)^{2k+1}(n+d)!} \leq \frac{(n+2k+1+l)!}{(l+1+n)^{2k+1}(n+l)!}.$$

This means that

$$\begin{aligned} LHS^{(1)} &\leq \frac{n!(n+2k+1+l)!}{(l+1+n)^{2k+1}(n+l)!(n+2k+1)!} \|f\|_{2k+1}^2 \\ &\leq \frac{(n+2k+1+l)^l}{(l+1+n)^{2k+1}} \|f\|_{2k+1}^2 = RHS^{(1)}, \end{aligned}$$

which completes the proof of (1).

(2) We begin the proof of (2) with an observation. Although the range of $T_{z_j}^{(2k+1)*}$ is contained in $L^2_{a,2k+1}(\mathbb{B}_n)$, it's easy to see that the image of an analytic polynomial under $T_{z_j}^{(2k+1)*}$ is still an analytic polynomial. This follows from the fact that

$$T_{z_j}^{(2k+1)*}(z^\alpha) = \frac{\alpha_j}{n+2k+1+|\alpha|} z^{\alpha-\varepsilon_j},\ 1 \leq j \leq n. \tag{2.1}$$

Therefore it belongs to $L^2_a(\mathbb{B}_n)$ and the $\mathrm{LHS}^{(2)}$ makes sense if f is an analytic polynomial. Specializing (2.1) to $k = 0$, one sees that

$$T_{z_j}^*(z^\alpha) = \frac{\alpha_j}{n+|\alpha|} z^{\alpha-\varepsilon_j},\ 1 \leq j \leq n. \tag{2.2}$$

Combining formulas (2.1), (2.2), we have

$$T^*_{z_j}(z^\alpha) - T^{(2k+1)*}_{z_j}(z^\alpha) = \frac{\alpha_j(2k+1)}{(n+|\alpha|)(n+2k+1+|\alpha|)} z^{\alpha-\varepsilon_j} = \frac{2k+1}{n+2k+1+|\alpha|} T^*_{z_j}(z^\alpha).$$

Thus, for any homogeneous analytic polynomial f with $d = \deg(f)$, one has that

$$T^*_{z_i}(f) - T^{(2k+1)*}_{z_i}(f) = \frac{2k+1}{n+2k+1+d} T^*_{z_i}(f).$$

This implies that

$$\begin{aligned} LHS^{(2)} &= \| \frac{1}{(d+n)^{k+1/2}} \frac{2k+1}{n+2k+1+d} T^*_{z_j}(f) \|^2 \\ &\le \frac{1}{(d+n)^{2k+1}} \frac{(2k+1)^2}{(n+2k+1+d)^2} \|f\|^2 \\ &= \frac{1}{(d+n)^{2k+1}} \frac{(2k+1)^2}{(n+2k+1+d)^2} \frac{n!}{(n+d)!} \frac{(n+2k+2+d)!}{(n+2k+2)!} \|f\|^2_{2k+2}. \end{aligned}$$

Using the same monotonicity argument as in (1), one shows that

$$\frac{(n+2k+2+d)!}{(n+2k+1+d)(d+n)^{2k+1}(n+d)!} \le \frac{(n+2k+2+l)!}{(n+2k+1+l)(l+n)^{2k+1}(n+l)!}.$$

Hence,

$$\begin{aligned} LHS^{(2)} &\le \frac{(2k+1)^2 n!(n+2k+2+l)!}{(n+2k+1+l)^2(l+n)^{2k+1}(n+l)!(n+2k+2)!} \|f\|^2_{2k+2} \\ &\le \frac{(n+2k+2+l)^l}{(l+n)^{2k+1}} \|f\|^2_{2k+2} = RHS^{(2)}, \end{aligned}$$

which completes the proof of the lemma. □

Using Proposition 2.2 and Lemma 2.3, we establish in the following proposition the necessary norm estimates for each term appearing in the infinite sum of Proposition 2.1.

Proposition 2.4. *For non-negative integers k, l and analytic polynomials $p, f \in \mathbb{C}[z_1, \cdots, z_n]$ satisfying $\partial_\alpha f(0) = 0$ for $|\alpha| < l$ and $m = \deg(p)$, we have the inequality*

$$\| \tfrac{1}{(N+1+n)^{k+1/2}} [M_{\partial_j R^k p} - M^*_{z_j} M_{R^{k+1}p}](f) \| \le \tfrac{(n+1)(n+2k+2+l)^{(l+n)/2} C(n,m)^{k+1}}{(l+n)^{k+1/2}} \|pf\|,$$

where $C(n, m)$ is the constant appearing in Proposition 2.2 which depends only on n, m.

Proof. The key idea of the proof is the following well-known identity (see e.g. [7])

$$\begin{aligned} \partial_j g - \overline{z_j} R g &= (1 - \sum_{i=1}^n |z_i|^2) \partial_j g + \sum_{i=1}^n z_i [\overline{z_i} \partial_j(g) - \overline{z_j} \partial_i(g)] \\ &= (1 - |z|^2) \partial_j g + \sum_{i=1, i\neq j}^n z_i L_{j,i}(g) \end{aligned} \tag{2.3}$$

for any smooth function g on $\mathbb{B}_n$.

Using the above identity with $g = R^k p$, we have

$$\begin{aligned}
&\|\frac{1}{(N+1+n)^{k+1/2}}[M_{\partial_j R^k p} - M_{z_j}^* M_{R^{k+1}p}](f)\| \\
\leq\ & \|\tfrac{1}{(N+1+n)^{k+1/2}}[M_{\partial_j R^k p} - T_{z_j}^{(2k+1)*} M_{R^{k+1}p}](f)\| + \|\tfrac{1}{(N+1+n)^{k+1/2}}(T_{z_j}^* - T_{z_j}^{(2k+1)*}) M_{R^{k+1}p}(f)\| \\
\leq\ & \tfrac{(n+l+2k+1)^{l/2}}{(l+1+n)^{k+1/2}} \|T^{(2k+1)}_{\partial_j R^k p - \overline{z_j} R^{k+1}p} f\|_{2k+1} + \tfrac{(n+2k+2+l)^{l/2}}{(l+n)^{k+1/2}} \|M_{R^{k+1}p}(f)\|_{2k+2} \\
\leq\ & \tfrac{(n+2k+2+l)^{l/2}}{(l+n)^{k+1/2}} [\|(1-|z|^2)\partial_j R^k(p)\, f\|_{2k+1} + \sum_{i=1, i\neq j}^{n} \|L_{j,i} R^k(p) f\|_{2k+1} + \|R^{k+1}(p) f\|_{2k+2}] \\
\leq\ & \tfrac{(n+2k+2+l)^{l/2}}{(l+n)^{k+1/2}} [n\sqrt{\tfrac{c_{2k+1} C(n,m)^{k+1}}{c_{2k}}} \|R^k(p) f\|_{2k} + \|R^{k+1}(p) f\|_{2k+2}] \\
\leq\ & \frac{(n+1)(n+2k+2+l)^{l/2} C(n,m)^{k+2} \sqrt{c_{2k+2}}}{(l+n)^{k+1/2}} \|pf\| \\
\leq\ & \frac{(n+1)(n+2k+2+l)^{(l+n)/2} C(n,m)^{k+2}}{(l+n)^{k+1/2}} \|pf\|.
\end{aligned}$$

The first inequality follows from the triangle inequality, while the second one is implied by Lemma 2.3, and that to the fourth line follows from formula (2.3) and the triangle inequality. Finally the inequalities of the second and third lines from the end follow from Proposition 2.2. This completes the proof of the proposition. □

We now prove the essential normality of $\mathcal{M} = [p]$ for $p \in \mathbb{C}[z_1, \cdots, z_n]$.

Theorem 2.5. *If $\mathcal{M} = [p]$ is the cyclic submodule of the Bergman space $L_a^2(\mathbb{B}_n)$ generated by an analytic polynomial $p \in \mathbb{C}[z_1, \cdots, z_n]$, then $\mathcal{M}$ is $p-$essentially normal for $p > n$.*

Proof. Suppose that $m = \deg(p)$ and fix l satisfying $n + l \geq 2C(n, m)$. Let

$$\mathscr{E}_l = \{f \in \mathbb{C}[z_1, z_2, \cdots, z_n] : \partial_\alpha f(0) = 0 \ for\ |\alpha| < l\}.$$

For any integer j with $1 \leq j \leq n$, define $D_j : p\mathscr{E}_l \subset L_a^2(\mathbb{B}_n) \to L_a^2(\mathbb{B}_n)$ by

$$D_j(pf) = \sum_{k=0}^{\infty} \frac{1}{(N+1+n)^{k+1/2}} [M_{\partial_j R^k p} - M_{z_j}^* M_{R^{k+1}p}](f), \ \ f \in \mathscr{E}_l.$$

By Proposition 2.4, D_j is a bounded operator.

Let P_l be the projection from $L_a^2(\mathbb{B}_n)$ to the closure $\mathcal{M}_l$ of $p\mathscr{E}_l$ in $\mathcal{M} = [p]$. Using Proposition 2.1, we have that for any polynomial $f \in \mathscr{E}_l$

$$\begin{aligned}
P_{\mathcal{M}^\perp} M_{z_j}^* P_l(pf) &= P_{\mathcal{M}^\perp} M_p M_{z_j}^*(f) + P_{\mathcal{M}^\perp} \frac{1}{(N+1+n)^{1/2}} D_j(pf) \\
&= P_{\mathcal{M}^\perp} \frac{1}{(N+1+n)^{1/2}} D_j(pf).
\end{aligned}$$

This means that $P_{\mathcal{M}^\perp} M_{z_j}^* P_l$ is in the Schatten $p-$class for $p > 2n$ by the fact that $\frac{1}{N+1+n}$ is in the Schatten $p-$class for $p > n$ as shown in [2] and D_j is bounded.

Since $\mathcal{M}_l$ is a finite codimensional subspace of $\mathcal{M}$, for any integer j with $1 \le j \le n$ we have $P_{\mathcal{M}^\perp} M_{z_j}^* P_{\mathcal{M}}$ is also in the Schatten $p-$class for $p > 2n$. By Lemma 2.1 in [17], one sees that $\mathcal{M}$ is $p-$essentially normal for $p > n$. $\square$

Remark 2.6. *Theorem 2.5 can be generalized to the vector-valued case with a slight modification. Let* $\mathbf{p} = (p_1, \cdots, p_r) \in \mathbb{C}[z_1, \cdots, z_n] \otimes \mathbb{C}^r$, *where each* p_i *is a polynomial with* $\deg(p_i) \le m$ *for some fixed* m, *and* $\mathcal{M} = [\mathbf{p}]$ *be the submodule of* $L_a^2(\mathbb{B}_n) \otimes \mathbb{C}^r$ *generated by* $\mathbf{p}$. *For* $1 \le j \le n$, *define* $D_j = (D_{j,1}, \cdots, D_{j,r}) : \mathbf{p}\mathscr{E}_l \to L_a^2(\mathbb{B}_n) \otimes \mathbb{C}^r$ *by*

$$D_{j,i}(p_i f) = \sum_{k=0}^{\infty} \frac{1}{(N+1+n)^{k+1/2}} [M_{\partial_j R^k p_i} - M_{z_j}^* M_{R^{k+1} p_i}](f), \; f \in \mathscr{E}_l.$$

Using an argument similar to that for Theorem 2.5, one sees that for any $f \in \mathscr{E}_l$, $P_{\mathcal{M}^\perp} M_{z_j}^* P_l(\mathbf{p}f) = P_{\mathcal{M}^\perp} \frac{1}{(N+1+n)^{1/2}} D_j(\mathbf{p}f)$. *Thus, one can obtain that* $P_{\mathcal{M}^\perp} M_{z_j}^* P_{\mathcal{M}} \in \mathcal{L}^p$ *for* $p > 2n$. *This means that the submodule* $[\mathbf{p}]$ *is* p*-essentially normal for* $p > n$.

3. Proof of Proposition 2.2

We will complete the proof of Proposition 2.2 in this section by proving an equivalent variant of it.

In what follows, we set $\Omega_r = \{z \in \mathbb{B}_n : |z| > r\}$ for $0 < r < 1$.

Proposition 2.2′. *For positive integers* n *and* m, *there is a positive constant* $C(n,m) > 1$ *such that for an analytic polynomial* $p \in \mathbb{C}[z_1, \cdots, z_n]$ *with degree* m, *the following inequalities hold for any analytic polynomial* f *and non-negative integers* i, j, k, l *with* $0 \le l \le k$, $1 \le i \ne j \le n$:

(1) $$\int_{\Omega_{\frac12}} |(R^l p)(z)\, f(z)|^2 (1-|z|^2)^{2k} dm(z) \le C(n,m)^{k+1} \int_{\mathbb{B}_n} |p(z) f(z)|^2 (1-|z|^2)^{2k-2l} dm(z);$$

(2) $$\int_{\Omega_{\frac12}} |(L_{j,i} p)(z)\, f(z)|^2 (1-|z|^2)^{2k+1} dm(z) \le C(n,m)^{k+1} \int_{\mathbb{B}_n} |p(z) f(z)|^2 (1-|z|^2)^{2k} dm(z);$$

(3) $$\int_{\Omega_{\frac12}} |(\partial_j p)(z)\, f(z)|^2 (1-|z|^2)^{2k+2} dm(z) \le C(n,m)^{k+1} \int_{\mathbb{B}_n} |p(z) f(z)|^2 (1-|z|^2)^{2k} dm(z).$$

Note that the constants c_t appearing in the statements of Proposition 2.2 are implicit here since integrals have replaced norms in these statements. With that observation it's easy to see that Proposition 2.2′ follows from Proposition 2.2. We use the following lemma to prove the other direction.

Lemma 3.1. *For a non-negative integer* t *and* $f \in L_{a,t}^2(\mathbb{B}_n)$, *we have*

$$\int_{\mathbb{B}_n} |f(z)|^2 (1-|z|^2)^t dm(z) \le 3^{t+1} \int_{\Omega_{\frac12}} |f(z)|^2 (1-|z|^2)^t dm(z).$$

Proof. We begin with the case $t = 0$. It's easy to see that
$$\int_{|z|<\frac{1}{2}} |z^\alpha|^2 dm(z) = (\frac{2}{3})^{2|\alpha|+2n} \int_{|z|<\frac{3}{4}} |z^\alpha|^2 dm(z).$$
Thus,
$$\int_{|z|<\frac{1}{2}} |z^\alpha|^2 dm(z) = \frac{(\frac{2}{3})^{2|\alpha|+2n}}{1-(\frac{2}{3})^{2|\alpha|+2n}} \int_{\frac{1}{2}<|z|<\frac{3}{4}} |z^\alpha|^2 dm(z) \leq 2 \int_{\frac{1}{2}<|z|<\frac{3}{4}} |z^\alpha|^2 dm(z).$$
Therefore, for each analytic function f on $\mathbb{B}_n$, it follows that
$$\int_{|z|<\frac{1}{2}} |f(z)|^2 dm(z) \leq 2 \int_{\frac{1}{2}<|z|<\frac{3}{4}} |f(z)|^2 dm(z).$$
For the general case $t \geq 0$, we have
$$\begin{aligned} &\int_{|z|<\frac{1}{2}} |f(z)|^2 (1-|z|^2)^t dm(z) \leq \int_{|z|<\frac{1}{2}} |f(z)|^2 dm(z) \\ \leq\ & 2\int_{\frac{1}{2}<|z|<\frac{3}{4}} |f(z)|^2 dm(z) \leq 3^{t+1} \int_{\frac{1}{2}<|z|<\frac{3}{4}} |f(z)|^2 (1-|z|^2)^t dm(z), \end{aligned}$$
which leads to the desired result. □

Now we show how to prove Proposition 2.2 from Proposition 2.2′.

By Lemma 3.1, clearly (1), (3) in Proposition 2.2 and Proposition 2.2′ are equivalent. Inequality (2) is not so obvious since $L_{j,i}(p)$ is not analytic in general. To avoid unnecessary complexity, we show that (2) and (3) of Proposition 2.2′ imply (2) of Proposition 2.2. In fact, (2) of Proposition 2.2′ implies that
$$c_{2k+1} \int_{\Omega_{\frac{1}{2}}} |(L_{j,i}p)(z)\, f(z)|^2 (1-|z|^2)^{2k+1} dv(z) \leq \frac{c_{2k+1} C(n,m)^{k+1}}{c_{2k}} \|p(z)f(z)\|_{2k}^2;$$
and using Lemma 3.1 and (3) of Proposition 2.2′ one shows that
$$\begin{aligned} & \|(L_{j,i}p)(z)\, f(z)\|_{2k+1}^2 - c_{2k+1} \int_{\Omega_{\frac{1}{2}}} |(L_{j,i}p)(z)\, f(z)|^2 (1-|z|^2)^{2k+1} dv(z) \\ =\ & c_{2k+1} \int_{|z|<\frac{1}{2}} |(L_{j,i}p)(z)\, f(z)|^2 (1-|z|^2)^{2k+1} \tfrac{dm(z)}{Vol(\mathbb{B}_n)} \\ \leq\ & c_{2k+1} \int_{|z|<\frac{1}{2}} 2\big[|\overline{z_j}(\partial_i p)(z)\, f(z)|^2 + |\overline{z_i}(\partial_j p)(z)\, f(z)|^2\,\big](1-|z|^2)^{2k+1} \tfrac{dm(z)}{Vol(\mathbb{B}_n)} \\ \leq\ & 4c_{2k+1} \int_{|z|<\frac{1}{2}} \big[\,|(\partial_i p)(z)\, f(z)|^2 + |(\partial_j p)(z)\, f(z)|^2\,\big](1-|z|^2)^{2k+2} \tfrac{dm(z)}{Vol(\mathbb{B}_n)} \\ \leq\ & 4\cdot 3^{2k+3} c_{2k+1} \int_{\Omega_{\frac{1}{2}}} [\,|(\partial_i p)(z)\, f(z)|^2 + |(\partial_j p)(z)\, f(z)|^2\,](1-|z|^2)^{2k+2} \tfrac{dm(z)}{Vol(\mathbb{B}_n)} \\ \leq\ & 8\cdot 3^{2k+3} c_{2k+1} C(n,m)^{k+1} \int_{\mathbb{B}_n} |p(z)\, f(z)|^2 (1-|z|^2)^{2k} \tfrac{dm(z)}{Vol(\mathbb{B}_n)} \\ \leq\ & \frac{c_{2k+1}(8\cdot 3^3 C(n,m))^{k+1}}{c_{2k}} \|p(z)f(z)\|_{2k}^2. \end{aligned}$$

Therefore, $\|(L_{j,i}p)(z)\, f(z)\|^2_{2k+1} \leq \frac{c_{2k+1}(217C(n,m))^{k+1}}{c_{2k}}\|p(z)f(z)\|^2_{2k}$, as desired, and we have shown that Propositions 2.2 and 2.2′ are equivalent.

The remainder of this section will be devoted to the proof of the weight norm estimates in Proposition 2.2′. The strategy of that is similar to the argument in [15]. However, we will give a complete proof, since in our proof we need to keep careful track of the constants. Let us begin with a local result in dimension one.

Lemma 3.2. *For a one-variable analytic polynomial $p \in \mathbb{C}[z]$ with $m \geq \deg(p)$, an integer l with $1 \leq l \leq m$ and an analytic function f on the complex plane $\mathbb{C}$, we have*

(1) $|\partial^l p(0)\, f(0)| \leq \frac{m!}{(m-l)!}\int_{\mathbb{T}} |pf| \frac{d\theta}{2\pi}$, where $\frac{d\theta}{2\pi}$ is the normalized Lebesgue measure on the unit circle $\mathbb{T}$.

(2) $r^l|\partial^l p(0)\, f(0)| \leq \frac{(l+2)m!}{2(m-l)!}\int_{r\mathbb{D}} |pf| \frac{dm(z)}{\pi r^2}$, where $\frac{dm(z)}{\pi r^2}$ is the normalized Lebesgue measure on the disk $r\mathbb{D}$.

Proof. (1) Without loss of generality, suppose $m = \deg(p)$ and

$$p(z) = z^u(z-a_1)\cdots(z-a_v)(z-b_1)\cdots(z-b_s),$$

where $u+v+s = m, |a_i| \geq 1, |b_i| < 1, b_i \neq 0$. It's easy to see that $|\partial^l p(0)| = 0$ if $l < u$. Moreover, for $l \geq u$ we have

$$|\partial^l p(0)| = |l! \sum_{\substack{\Lambda_1\subseteq\{1,2,\cdots,v\};\\ \Lambda_2\subseteq\{1,2,\cdots,s\};\\ |\Lambda_1|+|\Lambda_2|=m-l}} \prod_{i\in\Lambda_1, j\in\Lambda_2} a_i b_j| \leq l! \sum_{\substack{\Lambda_1\subseteq\{1,2,\cdots,v\};\\ \Lambda_2\subseteq\{1,2,\cdots,s\};\\ |\Lambda_1|+|\Lambda_2|=m-l}} |a_1\cdots a_v| \leq \frac{m!}{(m-l)!}|a_1\cdots a_v|.$$

Therefore,

$$\begin{aligned}
\int_{\mathbb{T}} |pf|\frac{d\theta}{2\pi} &= \int_{\mathbb{T}} |(z-a_1)\cdots(z-a_v)(z-b_1)\cdots(z-b_s)f|\frac{d\theta}{2\pi} \\
&= \int_{\mathbb{T}} |(z-a_1)\cdots(z-a_v)(1-\overline{b_1}z)\cdots(1-\overline{b_s}z)f|\frac{d\theta}{2\pi} \\
&\geq |a_1\cdots a_v||f(0)| \geq \frac{(m-l)!|\partial^l p(0)f(0)|}{m!}.
\end{aligned}$$

(2) For $r > 0$ and the analytic function f, let $f_r(z) = f(rz)$. Then we have

$$\begin{aligned}
\int_{r\mathbb{D}} |pf|\frac{dm(z)}{\pi r^2} &= \int_{0<r'<r}\int_{\theta} |p(r'e^{i\theta})f(r'e^{i\theta})|\frac{r'dr'd\theta}{\pi r^2} \\
&\geq \int_{0<r'<r} |\frac{2\pi(m-l)!}{m!}\partial^l p_{r'}(0)f_{r'}(0)|\frac{r'dr'}{\pi r^2} \\
&= \frac{2(m-l)!}{m!}|\partial^l p(0)f(0)|\int_{0<r'<r}\frac{r'^{l+1}dr'}{r^2} \\
&= \frac{2(m-l)!}{(l+2)m!}|r^l\,\partial^l p(0)f(0)|,
\end{aligned}$$

ending the proof of the lemma. □

We will establish the full inequalities in Proposition 2.2′ using the local result from the preceding lemma and the following Covering Lemma. We start by defining a special family of open subsets of $\mathbb{C}^n$.

Definition 3.3. *For any $a \in \mathbb{C}^n - \{0\}$, let P_a be the orthogonal projection from $\mathbb{C}^n$ onto the one-dimensional subspace $[a]$ generated by a, and $P_a^\perp$ be the orthogonal projection from $\mathbb{C}^n$ onto $\mathbb{C}^n \ominus [a]$. Given $\delta > 0$, define the neighborhood $Q_\delta(a)$ of a by*

$$Q_\delta(a) = \{z \in \mathbb{C}^n : |P_a(z) - a| < \delta, |P_a^\perp(z)| < \sqrt{\delta}\}.$$

Lemma 3.4. *Fixing $\frac{1}{4} < r < 1$ and $0 < c < \min\{\frac{r-\frac{1}{4}}{4}, \frac{1}{10}\}$, define $\delta(z) = c(1-|z|)$. For $z \in \Omega_r$, we have:*

(1) For any $z' \in Q_{\delta(z)}(z)$,

$$1 - 3c < \frac{1-|z'|^2}{1-|z|^2} < 1 + 2c;\ \frac{1}{3} < \frac{1-|z'|}{1-|z|} < 3;\ 1 - 4c < \frac{|z'|}{|z|}.$$

(2) $Q_{\delta(z)}(z) \subseteq \Omega_{r-4c} \subseteq \Omega_{\frac{1}{4}}$.

(3) There exists a constant $C = 200$ independent of z, r, c such that, if $z' \in Q_{\delta(z)}(z)$, then $Q_{\delta(z)}(z) \subset Q_{C\,\delta(z')}(z')$ and $Q_{\delta(z')}(z') \subset Q_{C\,\delta(z)}(z)$.

Proof. Using rotations in $\mathbb{C}^n$, without loss of generality we can suppose $z = (a, 0, 0, \cdots, 0)$ and $z' = (b_1, b_2, 0, \cdots, 0)$ with $0 < a < 1, 0 < b_2$.

(1)By the definition of $Q_{\delta(z)}(z)$, $|b_1 - a| < \delta(z)$ and $|b_2| < \sqrt{\delta(z)}$. This implies that

$$|b_1| < a + \delta(z) < a + \frac{1-a}{10} < 1.$$

Furthermore, using a direct computation one sees that

$$\frac{1-|z'|^2}{1-|z|^2} = 1 + \frac{|z|^2 - |z'|^2}{1-|z|^2} = 1 + \frac{a^2 - |b_1|^2}{1-|z|^2} - \frac{|b_2|^2}{1-|z|^2}$$

and

$$0 \le \frac{|a^2 - |b_1|^2|}{1-|z|^2} \le \frac{(a+|b_1|)|a-b_1|}{(1+|z|)(1-|z|)} < 2c,\ 0 \le \frac{|b_2|^2}{1-|z|^2} < c.$$

Therefore,

$$1 - 3c < \frac{1-|z'|^2}{1-|z|^2} < 1 + 2c.$$

This implies that

$$\frac{1}{3} < \frac{(1-3c)(1+|z|)}{1+|z'|} < \frac{1-|z'|}{1-|z|} < \frac{(1+2c)(1+|z|)}{1+|z'|} < 3. \tag{3.1}$$

Moreover, since $(1-4c)|z| < |b_1|$, we have $1 - 4c < \frac{|z'|}{|z|}$.

(2) From (1) it follows that $1 > |z'| > |z| - 4c \ge r - 4c \ge \frac{1}{4}$, as desired.

(3) For a point $w \in \mathbb{C}^n$, write $w = (w_1, w_2, w')$ with $w_1, w_2 \in \mathbb{C}, w' \in \mathbb{C}^{n-2}$. If $w = (w_1, w_2, w') \in Q_{\delta(z')}(z')$, then by Definition 3.3 and inequality (3.1) we have that $|w'| < \sqrt{\delta(z')} < \sqrt{3\delta(z)}$, and

$$(w_1, w_2) = u(b_1, b_2) + s(-b_2, \overline{b_1})$$

with $|s| < \sqrt{\frac{\delta(z')}{|b_1|^2+|b_2|^2}} \leq 4\sqrt{\delta(z')}$ and $|(u-1)(b_1,b_2)| < \delta(z')$. This means that

$$\begin{aligned} |w_1 - a| &= |ub_1 - sb_2 - a| \leq |(u-1)b_1| + |b_1 - a| + |sb_2| \\ &< \delta(z') + \delta(z) + 4\sqrt{\delta(z')\delta(z)} < 16\delta(z) \end{aligned}$$

and

$$\begin{aligned} |w_2| &= |ub_2 + s\overline{b_1}| \leq |(u-1)b_2| + |b_2| + |sb_1| \\ &< \delta(z') + \sqrt{\delta(z)} + 4\sqrt{\delta(z')} \leq 6\sqrt{3\delta(z)}. \end{aligned}$$

So, $Q_{\delta(z')}(z') \subset Q_{200\delta(z)}(z)$.

On the other hand, if $w = (w_1, w_2, w') \in Q_{\delta(z)}(z)$, by Definition 3.3 we have $|w'|, |w_2| < \sqrt{\delta(z)}$ and $|w_1 - a| < \delta(z)$. A direct computation shows that

$$(w_1, w_2) = \frac{w_1\overline{b_1} + w_2 b_2}{|b_1|^2 + |b_2|^2}(b_1, b_2) + \frac{w_2 b_1 - w_1 b_2}{|b_1|^2 + |b_2|^2}(-b_2, \overline{b_1}).$$

Since

$$\begin{aligned} & |\frac{w_1\overline{b_1} + w_2 b_2}{|b_1|^2 + |b_2|^2}(b_1, b_2) - (b_1, b_2)| \\ \leq\ & |\frac{(w_1 - a)\overline{b_1}}{|b_1|^2 + |b_2|^2}(b_1, b_2)| + |\frac{w_2 b_2}{|b_1|^2 + |b_2|^2}(b_1, b_2)| + |\frac{a\overline{b_1}}{|b_1|^2 + |b_2|^2}(b_1, b_2) - (b_1, b_2)| \\ \leq\ & 5\delta(z) + 4|a\overline{b_1} - |b_1|^2 - |b_2|^2| \leq 13\delta(z) \leq 39\delta(z'); \end{aligned}$$

and

$$|\frac{w_2 b_1 - w_1 b_2}{\sqrt{|b_1|^2 + |b_2|^2}}| \leq 8\sqrt{\delta(z)} \leq 8\sqrt{3\delta(z')},$$

it follows that we have $Q_{\delta(z')}(z') \subset Q_{200\delta(z)}(z)$ as desired. □

Proposition 3.5 (Covering Lemma). *Fix $r = \frac{1}{2}$, $c = \frac{1}{10\cdot 200^3}$ and define $\delta(z) = c(1 - |z|)$. Then there exists a countable set of points $\{z_s\}$ in Ω_r having the following properties:*

(i) $\Omega_r \subseteq \bigcup_s Q_{\delta(z_s)}(z_s)$ and $Q_{200^{-2}\delta(z_j)}(z_j) \cap Q_{200^{-2}\delta(z_s)}(z_s) = \emptyset$ if $j \neq s$.

(ii) $Q_{200^2\,\delta(z_s)}(z_s) \subseteq \Omega_{r-c}$, and no point belongs to more than $N(n) + 1$ of the sets $Q_{200^2\,\delta(z_s)}(z_s)$, where $N(n) = 200^{6n+6}$ depends only on the dimension n.

Proof. First we choose $\{z_s\}$ satisfying (i) by a classical method of harmonic analysis.

Set $\Gamma_1 = \{Q_{200^{-2}\delta(z)}(z) : z \in \Omega_r\}$. Let r_1 be the supremum of the radii $200^{-2}\delta(z)$ of the members $Q_{200^{-2}\delta(z)}(z)$ of Γ_1. Choose $z_1 \in \Omega_r$ with radius $200^{-2}\delta(z_1) > \frac{r_1}{2}$. Discard all the sets in Γ_1 that intersect $Q_{200^{-2}\delta(z_1)}(z_1)$, and denote the remaining collection by Γ_2. Let r_2 be the supremum of the radii of the members of Γ_2 and choose z_2 with radius $200^{-2}\delta(z_2) > \frac{r_2}{2}$. After, discarding all the sets in Γ_2 that intersect $Q_{200^{-2}\delta(z_2)}(z_2)$, denote the remaining collection by Γ_3, and continue inductively. One sees that the process will continue through the natural numbers. We thus get a sequence $\{z_s\}$ such that $Q_{200^{-2}\delta(z_j)}(z_j) \cap Q_{200^{-2}\delta(z_s)}(z_s) = \emptyset$ if $j \neq s$.

If some $Q_{200^{-2}\delta(z)}(z) \in \Gamma_1$ was discarded at the $j-$th stage, then $Q_{200^{-2}\delta(z)}(z) \cap Q_{200^{-2}\delta(z_j)}(z_j) \neq \emptyset$. Fixing a point z' in the intersection, by Lemma 3.4 (3) we have

$$Q_{200^{-2}\delta(z)}(z) \subseteq Q_{200^{-1}\delta(z')}(z') \subseteq Q_{\delta(z_j)}(z_j).$$

Therefore,

$$\Omega_r \subseteq \bigcup_{z\in\Omega_r} Q_{200^{-2}\delta(z)}(z) \subseteq \bigcup_{s} Q_{\delta(z_s)}(z_s).$$

This means that the sequence $\{z_s\}$ satisfies (i).

Now we show that the sequence $\{z_s\}$ satisfies (ii). From Lemma 3.4 (2), clearly $Q_{200^2\,\delta(z_s)}(z_s) \subseteq \Omega_{r-c}$. For any $z \in \Omega_{r-c}$, let

$$\Lambda_z = \{j : z \in Q_{200^2\delta(z_j)}(z_j)\} \subseteq \mathbb{N}.$$

Using Lemma 3.4 (1) and (2), one sees that

$$Q_{200^2\delta(z_j)}(z_j) \subseteq Q_{200^3\delta(z)}(z); \qquad \frac{\delta(z_j)}{3} < \delta(z) < 3\delta(z_j) \quad \forall j \in \Lambda_z.$$

By the fact that $Q_{200^{-2}\delta(z_j)}(z_j) \cap Q_{200^{-2}\delta(z_s)}(z_s) = \emptyset, \forall j, s \in \Lambda_z, j \neq s$, and

$$\cup_{j\in\Lambda_z} Q_{200^{-3}\delta(z)}(z_j) \subseteq \cup_{j\in\Lambda_z} Q_{200^{-2}\delta(z_j)}(z_j) \subseteq Q_{200^3\delta(z)}(z),$$

we have $|\Lambda_z| \leq \frac{Vol(Q_{200^3\delta(z)}(z))}{Vol(Q_{200^{-3}\delta(z)}(z))} = 200^{6n+6}$, which establishes (ii). $\Box$

Now we turn to the proof of Proposition 2.2$'$. Here we use the same notation as in Proposition 3.5.

Proof of Proposition 2.2$'$ (2)

We begin with a local result, i.e., an inequality which holds on $Q_{\delta(z)}(z)$ with $z = (a, 0, 0, \cdots, 0)$. Obviously, $L_{j,i} \neq 0$ only if $i = 1, j > 1$ or $i > 1, j = 1$; and in these cases $L_{j,i} = \overline{a}\partial_j$ or $L_{j,i} = -\overline{a}\partial_i$, respectively.

We consider the complex tangential derivative ∂_2 first. For a point $w \in \mathbb{C}^n$, write $w = (z_1, z_2, z')$ with $z_1, z_2 \in \mathbb{C}, z' \in \mathbb{C}^{n-2}$. For any z_1, z' satisfying $|z_1 - a| < \delta(z), |z'| < \sqrt{0.5\delta(z)}$, we have $w \in Q_{\delta(z)}(z)$ if $|z_2| < \sqrt{0.5\delta(z)}$.

Using Lemma 3.2 one shows that, if $|z_1 - a| < \delta(z)$ and $|z'| < \sqrt{0.5\delta(z)}$, then

$$|\sqrt{0.5\delta(z)}\partial_2 p(z_1, 0, z')f(z_1, 0, z')| \leq 2m \int_{|z_2|<\sqrt{0.5\delta(z)}} |p(z_1, z_2, z')f(z_1, z_2, z')| \frac{dm(z_2)}{0.5\pi\delta(z_2)}.$$

Therefore,

$$\begin{aligned}
&|\sqrt{0.5\delta(z)}\partial_2 p(a, 0, 0)f(a, 0, 0)| \\
\leq\ & \int_{|z_1-a|<\delta(z),|z'|<\sqrt{0.5\delta(z)}} |\sqrt{0.5\delta(z)}\partial_2 p(z_1, 0, z')f(z_1, 0, z')| \frac{dm(z_1)}{\pi\delta(z)^2} \frac{dm(z')}{Vol\{|z'| < \sqrt{0.5\delta(z)}\}} \\
\leq\ & 2^n m \int_{w\in Q_{\delta(z)}(z)} |p(w)f(w)| \frac{dm(w)}{Vol(Q_{\delta(z)}(z))}.
\end{aligned}$$

Using Hölder's inequality, we have

$$|\sqrt{0.5\delta(z)}\partial_2 p(a, 0, 0)f(a, 0, 0)|^2 \leq 2^{2n}m^2 \int_{w\in Q_{\delta(z)}(z)} |p(w)f(w)|^2 \frac{dm(w)}{Vol(Q_{\delta(z)}(z))}.$$

The same argument is also valid for ∂_j, $1 < j \leq n$. This implies that

$$|\nabla_T p(z) f(z)|^2(1-|z|) \leq \frac{2^{2n+1}m^2}{c} \int_{w\in Q_{\delta(z)}(z)} |p(w)f(w)|^2 \frac{dm(w)}{Vol(Q_{\delta(z)}(z))}.$$

The expression $|\nabla_T p(z)|$ is called the tangential gradient of p at z (see e.g. [21, Section 7.6]) with the definition

$$|\nabla_T p(z)| = max\{|\sum_{i=1}^n u_i \partial_i p(z)| : u \in \partial\mathbb{B}_n, u \perp z\}.$$

Using rotation, the above inequality is valid for any $z \in \Omega_r$ with $r = \frac{1}{2}$. This means that for any $z \in \Omega_{1/2}, 1 \leq i \neq j \leq n$, we have

$$|L_{j,i} p(z) f(z)|^2(1-|z|) \leq \frac{2^{2n+1}m^2}{c} \int_{w\in Q_{\delta(z)}(z)} |p(w)f(w)|^2 \frac{dm(w)}{Vol(Q_{\delta(z)}(z))}.$$

Therefore, for $1 \leq i \neq j \leq n$ one sees that

$$\begin{aligned}
& \int_{z\in Q_{\delta(z_s)}(z_s)} |L_{j,i} p(z) f(z)|^2 (1-|z|^2)^{2k+1} dm(z) \\
\leq\ & 2\int_{z\in Q_{\delta(z_s)}(z_s)} |L_{j,i} p(z) f(z)|^2 (1-|z|)(1-|z|^2)^{2k} dm(z) \\
\leq\ & \tfrac{2^{2n+2}m^2}{c}(1+2c)^{2k}(1-|z_s|^2)^{2k} \textstyle\int_{z\in Q_{\delta(z_s)}(z_s)} \big[\int_{w\in Q_{\delta(z)}(z)} |p(w)f(w)|^2 \frac{dm(w)}{Vol(Q_{\delta(z)}(z))}\big] dm(z) \\
\leq\ & \tfrac{2^{2n+2}m^2(1+2c)^{2k}}{c}(1-|z_s|^2)^{2k} \textstyle\int_{z\in Q_{\delta(z_s)}(z_s)} \big[\int_{w\in Q_{200\delta(z_s)}(z_s)} |p(w)f(w)|^2 \frac{dm(w)}{Vol(Q_{\delta(z)}(z))}\big] dm(z) \\
\leq\ & \frac{3^{n+1}2^{2n+2}m^2(1+2c)^{2k}}{c(1-3c)^{2k}} \int_{w\in Q_{200\delta(z_s)}(z_s)} |p(w)f(w)|^2 (1-|w|^2)^{2k} dm(w).
\end{aligned}$$

By Covering Lemma 3.5, we have

$$\begin{aligned}
& \int_{z\in\Omega_{1/2}} |L_{j,i} p(z) f(z)|^2 (1-|z|^2)^{2k+1} dm(z) \\
\leq\ & \textstyle\sum_s \int_{z\in Q_{\delta(z_s)}} |L_{i,j} p(z) f(z)|^2 (1-|z|^2)^{2k+1} dm(z) \\
\leq\ & \textstyle\sum_s \frac{3^{n+1}2^{2n+2}m^2(1+2c)^{2k}}{c(1-3c)^{2k}} \int_{z\in Q_{200\delta(z_s)}(z_s)} |p(z)f(z)|^2 (1-|z|^2)^{2k} dm(z) \\
\leq\ & \tfrac{3^{n+1}2^{2n+2}m^2(1+2c)^{2k}}{c(1-3c)^{2k}} N(n) \textstyle\int_{\mathbb{B}_n} |p(z)f(z)|^2 (1-|z|^2)^{2k} dm(z) \\
\leq\ & (24^{n+1}m^2N(n)/c)^{k+1} \textstyle\int_{\mathbb{B}_n} |p(z)f(z)|^2 (1-|z|^2)^{2k} dm(z),
\end{aligned}$$

as desired.

To prove inequality (1), the following lemma is needed.

Lemma 3.6. *For any smooth function f on the complex plane $\mathbb{C}$,*

$$R^l f = \sum_{j=1}^l a_j^{(l)} z^j \partial^j f$$

with $|a_j^{(l)}| < (j+1)^l$.

Proof. We prove the lemma by induction on l. Clearly it holds in the case $l = 1$. Suppose the inequality holds for the coefficients for $l = s$. For $l = s+1$, we have

$$\begin{aligned} R^{s+1} f &= z\partial(\sum_{j=1}^{s} a_j^{(s)} z^j \partial^j f) \\ &= z(\sum_{j=1}^{s} a_j^{(s)} j z^{j-1} \partial^j f + \sum_{j=1}^{s} a_j^{(s)} z^j \partial^{j+1} f) \\ &= \sum_{j=1}^{s+1} (j a_j^{(s)} + a_{j-1}^{(s)}) z^j \partial^j f, \end{aligned}$$

where we are assuming that $a_0^{(s)} = a_{s+1}^{(s)} = 0$. By the induction hypothesis $|a_j^{(s)}| < (j+1)^s$, one sees that

$$|a_j^{(s+1)}| = |j a_j^{(s)} + a_{j-1}^{(s)}| < (j+1)^{s+1},$$

which completes the proof of the lemma. $\square$

Now we return to prove inequality (1).

Proof of Proposition 2.2′ (1)

We first reduce the question to the case of dimension one. Indeed, define the slice function $g_\xi(z) = g(\xi z)$ for $g \in C(\overline{\mathbb{B}_n})$ and $\xi \in \partial\mathbb{B}_n$, $z \in \mathbb{D}$. Using Propositions 1.4.3 and Proposition 1.4.7(1) in [19], we have that for $g \in C(\overline{\mathbb{B}_n})$

$$\begin{aligned} \int_{\mathbb{B}_n} g dm &= 2nVol(\mathbb{B}_n) \int_{r\in[0,1]} r^{2n-1} dr \int_{\xi\in\partial\mathbb{B}_n} g(r\xi) d\sigma(\xi) \\ &= 2nVol(\mathbb{B}_n) \int_{r\in[0,1]} r^{2n-1} dr \int_{\xi\in\partial\mathbb{B}_n} d\sigma(\xi) \int_{\theta\in(-\pi,\pi]} g(re^{i\theta}\xi) \frac{d\theta}{2\pi} \\ &= \frac{1}{2\pi} \int_{\xi\in\partial\mathbb{B}_n} dm(\xi) \int_{z\in\mathbb{D}} g_\xi(z) |z^{n-1}|^2 dm(z), \end{aligned} \tag{3.2}$$

where $d\sigma(\xi) = \frac{dm(\xi)}{Vol(\partial\mathbb{B}_n)} = \frac{dm(\xi)}{2nVol(\mathbb{B}_n)}$ is the normalized Lebesgue measure on $\partial\mathbb{B}_n$.

Noticing that $R_z(p_\xi(z)) = (Rp)_\xi(z)$, where R_z is the radial derivative in the one variable z, by formula (3.2) we have

$$\begin{aligned} &\int_{\mathbb{B}_n} |(R^k p)(z)\, f(z)|^2 (1-|z|^2)^{2k} dm(z) \\ = &\frac{1}{2\pi} \int_{\xi\in\partial\mathbb{B}_n} \Big[\int_{z\in\mathbb{D}} |R_z^k(p_\xi(z))\, f_\xi(z) z^{n-1}|^2 (1-|z|^2)^{2k} dm(z) \Big] dm(\xi); \\ &\int_{\mathbb{B}_n} |p(z)\, f(z)|^2 dm(z) \\ = &\frac{1}{2\pi} \int_{\xi\in\partial\mathbb{B}_n} \Big[\int_{z\in\mathbb{D}} |p_\xi(z)\, f_\xi(z) z^{n-1}|^2 dm(z) \Big] dm(\xi). \end{aligned}$$

So, it suffices to show the inequality involving one variable functions.

Now we use the Covering Lemma to show the inequality on $\mathbb{C}$ for ∂^j, $1 \le j \le \min(m, l)$. In this case, the covering domains in Proposition 3.5 degenerate to disks

with radii $\delta(z)$. The same argument as in the proof of Proposition 2.2′ (2) shows that for $1 \le j \le min(m,l)$, one has that for $z \in \mathbb{D}_{\frac{1}{2}} = \{w \in \mathbb{D} : |w| > \frac{1}{2}\}$

$$|\partial^j p(z) f(z)|^2 \delta^{2j}(z) \le [\frac{(j+2)m!}{2(m-j)!}]^2 \int_{w\in Q_{\delta(z)}(z)} |p(w)f(w)|^2 \frac{dm(w)}{Vol(Q_{\delta(z)}(z))}.$$

This implies that if $1 \le j \le min(m,l) \le k$, then we have

$$\begin{aligned}
&\int_{z\in Q_{\delta(z_s)(z_s)}} |\partial^j p(z) f(z)|^2 (1-|z|^2)^{2k} dm(z)\\
\le\ & \frac{3^2(m+1)!^2 2^{2j}(1+2c)^{2k-2j}}{c^{2j}(1-3c)^{2k-2j}} \int_{w\in Q_{200\delta(z_s)}(z_s)} |p(w)f(w)|^2(1-|w|^2)^{2k-2j} dm(w),
\end{aligned}$$

and hence

$$\begin{aligned}
&\int_{z\in\mathbb{D}_{\frac{1}{2}}} |(\partial^j p)(z)\, f(z)|^2(1-|z|^2)^{2k} dm(z)\\
\le\ & \frac{3^2(m+1)!^2 2^{2j}(1+2c)^{2k-2j}}{c^{2j}(1-3c)^{2k-2j}} N(n) \int_{\mathbb{D}} |p(z)f(z)|^2(1-|z|^2)^{2k-2j} dm(z)\\
\le\ & (12^2(m+1)!^2 N(n)/c^2)^{k+1} \int_{\mathbb{D}} |p(z)f(z)|^2(1-|z|^2)^{2k-2l} dm(z).
\end{aligned}$$

Using Lemma 3.6 we show that for the polynomial p with $m = \deg(p)$

$$|R^l p| = |\sum_{j=1}^{\min\{l,m\}} a_j^{(l)} z^j \partial^j p| \le (m+1)^l \sum_{j=1}^{\min\{l,m\}} |\partial^j p|.$$

Therefore, one has

$$\begin{aligned}
&\int_{\{z\in\mathbb{D}:|z|>\frac{1}{2}\}} |(R^l p)(z)\, f(z)|^2(1-|z|^2)^{2k} dm(z)\\
\le\ & (12^2 m^2(m+1)(m+1)!^2 N(n)/c^2)^{k+1} \int_{\mathbb{D}} |p(z)f(z)|^2(1-|z|^2)^{2k-2l} dm(z),
\end{aligned}$$

completing the proof of (1). □

It remains to prove (3). One can prove it using the above methods or it can be shown directly from Proposition 2.2′ (1)(2) as follows.

Proof of Proposition 2.2′ (3)

By equation (2.3), we have $|z|^2\partial_j p = \overline{z_j} Rp + \sum_{i=1,i\neq j}^n z_i L_{j,i} p$, which implies that

$$|\partial_j p| \le 4|Rp| + \sum_{i\neq j} 4|L_{j,i}p|$$

for $|z| > 1/2$. Combing this inequality with Proposition 2.2′ $(1),(2)$ shows the desired result. □

4. Further discuss

4.1. **The weighted Bergman space $L_a^2(\mu_p)$.** For $p \in \mathbb{C}[z_1, \cdots, z_n]$, let $L^2(\mu_p)$ be the Hilbert space consisting of functions having the property that $\int_{\mathbb{B}_n} |f|^2 d\mu_p < \infty$, where μ_p is the measure on $\mathbb{B}_n$ defined by $d\mu_p = |p|^2 dm$, and let $L_a^2(\mu_p)$ be the weighted Bergman space consisting of the analytic functions in $L^2(\mu_p)$. Little is known about this natural analytic function space. In what follows, we show some elementary properties of $L_a^2(\mu_p)$ using the methods and results in Section 3.

Lemma 4.1. *For a polynomial $p \in \mathbb{C}[z_1, \cdots, z_n]$ with $m = \deg(p)$, we have for any $f \in L_a^2(\mu_p)$ that*

$$\int_{\mathbb{B}_n} |f_r|^2 |p|^2 dm \le 2^{2(m+n-1)} \int_{\mathbb{B}_n} |f|^2 |p|^2 dm, \quad \text{if } \frac{1}{2} < r < 1,$$

where $f_r(z) = f(rz)$ for $z \in \mathbb{B}_n$.

Proof. Firstly we show the inequality in the case of one dimension as follows.

For each polynomial g with $m = \deg(g)$, suppose

$$g(z) = z^u (z - a_1) \cdots (z - a_v)(z - b_1) \cdots (z - b_s),$$

where $u + v + s = m, |a_i| \ge 1, |b_i| < 1, b_i \ne 0$. Let

$$\widetilde{g}(z) = (z - a_1) \cdots (z - a_v)(1 - \overline{b_1} z) \cdots (1 - \overline{b_s} z).$$

By Lemma 2.1 in [14], one sees that $\frac{\widetilde{g}(z)}{\widetilde{g}(rz)} \le 2^m$ for $\frac{1}{2} < r < 1, |z| \le 1$. This implies that for $h \in A(\mathbb{D})$ and $\frac{1}{2} < r < 1$, we have

$$\begin{aligned} \int_{\mathbb{T}} |g(e^{i\theta}) h(re^{i\theta})|^2 \frac{dm(\theta)}{2\pi} &= \int_{\mathbb{T}} |\widetilde{g}(e^{i\theta}) h(re^{i\theta})|^2 \frac{dm(\theta)}{2\pi} \le 2^{2m} \int_{\mathbb{T}} |\widetilde{g}(re^{i\theta}) h(re^{i\theta})|^2 \frac{dm(\theta)}{2\pi} \\ &\le 2^{2m} \int_{\mathbb{T}} |\widetilde{g}(e^{i\theta}) h(e^{i\theta})|^2 \frac{dm(\theta)}{2\pi} = 2^{2m} \int_{\mathbb{T}} |g(e^{i\theta}) h(e^{i\theta})|^2 \frac{dm(\theta)}{2\pi}. \end{aligned}$$

Therefore, for $f \in L_a^2(\mu_p)$ and $\frac{1}{2} < r < 1$, one has

$$\begin{aligned} &\int_{\mathbb{D}} |p(z) f(r\, z)|^2 \frac{dm(z)}{\pi} = \int_{0<r'<1} \Big[\int_{\mathbb{T}} |p(r'\, e^{i\theta}) f(r\, r'\, e^{i\theta})|^2 \frac{dm(\theta)}{2\pi} \Big] 2rdr \\ \le\ & 2^{2m} \int_{0<r'<1} \Big[\int_{\mathbb{T}} |p(r'\, e^{i\theta}) f(r'\, e^{i\theta})|^2 \frac{dm(\theta)}{2\pi} \Big] 2rdr = 2^{2m} \int_{\mathbb{D}} |p(z) f(z)|^2 \frac{dm(z)}{\pi}, \end{aligned}$$

which establishes the inequality in the case of one dimension.

Now we prove the general case by a slice argument as in formula (3.2). Indeed, we have that

$$\begin{aligned} &\int_{\mathbb{B}_n} |f_r|^2 |p|^2 dm = \frac{1}{2\pi} \int_{\xi \in \partial \mathbb{B}_n} dm(\xi) \int_{z \in \mathbb{D}} |f(\xi r z)|^2 |z^{n-1} p(\xi z)|^2 dm(z) \\ \le\ & \frac{2^{2(m+n-1)}}{2\pi} \int_{\xi \in \partial \mathbb{B}_n} dm(\xi) \int_{z \in \mathbb{D}} |f(\xi z)|^2 |z^{n-1} p(\xi z)|^2 dm(z) = 2^{2(m+n-1)} \int_{\mathbb{B}_n} |f|^2 |p|^2 dm, \end{aligned}$$

which completes the proof. $\square$

Lemma 4.2. *The weighted Bergman space $L_a^2(\mu_p)$ is complete.*

Proof. It suffices to show that $L_a^2(\mu_p)$ is a closed subspace of $L^2(\mu_p)$. That is, if a sequence $\{f_n\}$ in $L_a^2(\mu_p)$ converges to f in the norm of $L^2(\mu_p)$, then f is equal a.e. to an analytic function on the unit ball. Choose a multi-index α such that $|\alpha| = \deg(p)$ and $\partial^\alpha p$ is a nonzero constant. Using the above lemma and Proposition 2.2(3), we have for any $\frac{1}{2} < r < 1$, that

$$\begin{aligned} & |\partial^\alpha p|^2 \int_{\mathbb{B}_n} |f_n(r\,z) - f_l(r\,z)|^2 (1-|z|^2)^{2|\alpha|} dm(z) \\ \leq \; & c_{2|\alpha|} \prod_{m=1}^{|\alpha|} C(n,m)^{1+|\alpha|-m} \int_{\mathbb{B}_n} |f_n(r\,z) - f_l(r\,z)|^2 |p(z)|^2 dm(z) \\ \leq \; & 2^{2(|\alpha|+n-1)} c_{2|\alpha|} \prod_{m=1}^{|\alpha|} C(n,m)^{1+|\alpha|-m} \int_{\mathbb{B}_n} |f_n(z) - f_l(z)|^2 |p(z)|^2 dm(z) \to 0 \end{aligned}$$

as $n, l \to \infty$, where $C(n,m)$ is the constant appearing in Proposition 2.2. This implies that the sequence f_n is pointwise convergent to an analytic function g. Noticing that f_n is also pointwise convergent to f outside the zero measure set $Z(p) \cap \mathbb{B}_n$, we have $f = g\,a.e.$, which completes the proof. □

Lemma 4.3. *The polynomial ring $\mathbb{C}[z_1, \cdots, z_n]$ is dense in $L_a^2(\mu_p)$.*

Proof. Let M be the closure of $\mathbb{C}[z_1, \cdots, z_n]$ in $L_a^2(\mu_p)$. Obviously, for each $g \in A(\mathbb{B}_n)$, we have $g \in M$. For any $f \in L_a^2(\mu_p)$, set $f_n(z) = f((1-\frac{1}{n})z)$. By Lemma 4.1 the sequence f_n is uniformly bounded in $L_a^2(\mu_p)$. So, there exists a subsequence f_{n_k} which is weakly convergent to some function $g \in L_a^2(\mu_p)$. Clearly $g \in M$. Moreover, for each $z \in \mathbb{B}_n$, by the proof for the above lemma, the point evaluation at z is a bounded functional in the Hilbert space $L_a^2(\mu_p)$. This implies that $f_{n_k}(z) \to g(z)$ for each $z \in \mathbb{B}_n$. Thus, $g = f$ and hence $f \in M$. This means that the closure $M = L_a^2(\mu_p)$. □

We summarize the results in this subsection in the following.

Theorem 4.4. *Let $p \in \mathbb{C}[z_1, \cdots, z_n]$. Set $d\mu_p = |p|^2 dm$ and*

$$L_a^2(\mu_p) = \{f \in L^2(\mu_p),\ f \text{ holomorphic on } \mathbb{B}_n\}.$$

Then $L_a^2(\mu_p)$ is a reproducing kernel Hilbert space on $\mathbb{B}_n$, which defines a p-essentially normal Hilbert module whose essential spectrum equals $\partial\mathbb{B}_n$. Moreover, $\mathbb{C}[z_1, \cdots, z_n]$ is dense in $L_a^2(\mu_p)$. And $L_a^2(\mu_p) \subset L_{a,t}^2(\mathbb{B}_n)$ for $t \geq 2\deg(p)$.

Proof. Consider the operator $\mathcal{I} : L_a^2(\mu_p) \to L_a^2(\mathbb{B}_n)$ defined by

$$\mathcal{I}(p) = pf.$$

This natural embedding map $\mathcal{I}$ is an isometrical module isomorphism from $L_a^2(\mu_p)$ to the image $\mathrm{ran}(\mathcal{I})$. Clearly the submodule $[p] \subseteq \mathrm{ran}(\mathcal{I})$. Furthermore, by Lemma 4.3, each of $[p]$ and $\mathrm{ran}(\mathcal{I})$ is the closure of the ideal $p\mathbb{C}[z_1, \cdots, z_n]$. This means that $[p] = \mathrm{ran}(\mathcal{I})$. Hence, by Theorem 2.5 one sees that $L_a^2(\mu_p)$ is essentially normal, which is a result analogous to the basic result for the Bergman space. □

Moreover, we also have obtained a somewhat surprising result in function theory since $[p] = \mathrm{ran}(\mathcal{I})$.

Corollary 4.5. *For any analytic function $f \in L^2_a(\mathbb{B}_n)$, one has that $f \in [p]$ if and only if $f = ph$ for some analytic function h on $\mathbb{B}_n$.*

4.2. **Quotient Modules.** Let $p \in \mathbb{C}[z_1, \cdots, z_n], \mathcal{M}_p = [p] \subseteq L^2_a(\mathbb{B}_n)$ be the cyclic submodule generated by p, $\mathscr{Q}_p$ be the quotient module defined by the short exact sequence

$$0 \longrightarrow \mathcal{M}_p \longrightarrow L^2_a(\mathbb{B}_n) \longrightarrow \mathscr{Q}_p \longrightarrow 0,$$

and Q_f be the compression of M_f on $L^2_a(\mathbb{B}_n)$ to $\mathscr{Q}_p$ for $f \in H^\infty(\mathbb{B}_n)$. Then the map $f \to Q_f$ for $f \in \mathbb{C}[z_1, \cdots, z_n]$ defines the module action of $\mathbb{C}[z_1, \cdots, z_n]$ on $\mathscr{Q}_p$.

Let $\mathscr{T}(\mathscr{Q}_p)$ be the C^*-subalgebra of $\mathscr{L}(\mathscr{Q}_p)$ generated by $\{Q_f : f \in \mathbb{C}[z_1, \cdots, z_n]\}$ and $\mathcal{K}(\mathscr{Q}_p)$ be the ideal of compact operators on $\mathscr{Q}_p$. From Theorem 2.5 and Lemma 2.1 in [17] or the related result in [3, 11, 16, 18], it follows that all the operators Q_f are essentially normal, or $[Q_f, Q_g^*] \in \mathcal{K}(\mathscr{Q}_p)$ for $f, g \in \mathbb{C}[z_1, \cdots, z_n]$, and hence $\mathscr{T}(\mathscr{Q}_p)/\mathcal{K}(\mathscr{Q}_p)$ is a commutative C^*-algebra. This means that it's isometrically isomorphic to $C(X_p)$ for some compact metrizable space X_p. Using the image of the n-tuple $(Q_{z_1}, \cdots, Q_{z_n})$ in $\mathscr{T}(\mathscr{Q}_p)/\mathcal{K}(\mathscr{Q}_p)$, we can identify X_p as a subset of $\mathbb{C}^n$. Moreover, since $\sum_{i=1}^n Q^*_{z_i} Q_{z_i} \leq I$, one sees that $X_p \subseteq \mathit{clos}\, \mathbb{B}_n$. In fact, we have the following partial characterization of X_p.

Proposition 4.6. *For $p \in \mathbb{C}[z_1, \cdots, z_n]$, we have*

$$\mathit{clos}\{Z(p) \cap \mathbb{B}_n\} \cap \partial\mathbb{B}_n \subseteq X_p \subseteq Z(p) \cap \partial\mathbb{B}_n.$$

Note that a point z_0 is in $Z(p) \cap \partial\mathbb{B}_n$ and not in $\mathit{clos}\{Z(p) \cap \mathbb{B}_n\}$ only when the component of $Z(p)$ containing z_0 is "tangent" to $\mathbb{B}_n$ in some sense.

Proof. For $f \in \mathbb{C}[z_1, \cdots, z_n]$, we can write

$$M_f = S_f \oplus Q_f + K,$$

where $K \in \mathcal{K}(L^2_a(\mathbb{B}_n))$. Since the C^*- algebra generated by $\{M_f : f \in \mathbb{C}[z_1, \cdots, z_n]\}$ contains $\mathcal{K}(L^2_a(\mathbb{B}_n))$ and $\mathscr{T}(L^2_a(\mathbb{B}_n))/\mathcal{K}(L^2_a(\mathbb{B}_n)) \cong C(\partial\mathbb{B}_n)$, we have a $*-$ homomorphism from $C(\partial\mathbb{B}_n)$ to $C(X_p)$. It follows that

$$X_p \subseteq \sigma_e\{M_{z_1}, \cdots, M_{z_n}\} = \partial\mathbb{B}_n,$$

where σ_e denotes the joint essential spectrum.

If $z_0 = (z^0_1, \cdots, z^0_n) \in \partial\mathbb{B}_n$ such that $p(z_0) \neq 0$, then the ideal in $\mathbb{C}[z_1, \cdots, z_n]$ generated by $\{z_1 - z^0_1, \cdots, z_n - z^0_n, p\}$ equals $\mathbb{C}[z_1, \cdots, z_n]$. Therefore, there exist polynomials $\{q_i\}_{i=1}^{n+1}$ such that

$$\sum_{i=1}^n q_i(z)(z_i - z^0_i) + q_{n+1}(z)p(z) \equiv 1.$$

This implies that $\sum_{i=1}^n Q_{q_i} Q_{z_i - z^0_i} = I_{\mathscr{Q}_p}$, or z_0 is not in the joint essential spectrum of the n-tuple $\{Q_{z_1}, \cdots, Q_{z_n}\}$ and $z_0 \notin X_p$.

Suppose $w_0 = (w^0_1, \cdots, w^0_n) \in \partial\mathbb{B}_n$ such that there exists $\{w_k\}_{k=1}^\infty \subseteq Z(p) \cap \mathbb{B}_n$ and $w_k \to w_0$. Let $\{\xi_k\}$ be unit vectors in $L^2_a(\mathbb{B}_n)$ such that $M^*_f \xi_k = \overline{f(w_k)}\xi_k$ for

$f \in \mathbb{C}[z_1, \cdots, z_n]$ and $k \in \mathbb{N}$. It's well known that ξ_k is weakly convergent to 0 since $\overline{w_0}$ is not a joint eigenvalue of the $n-$tuple $(M_{z_1}^*, \cdots, M_{z_n}^*)$. Since

$$\langle \xi_k, pf \rangle_{L_a^2(\mathbb{B}_n)} = \langle M_p^* \xi_k, f \rangle_{L_a^2(\mathbb{B}_n)} = \overline{p(\xi_k)} \langle \xi_k, f \rangle_{L_a^2(\mathbb{B}_n)} = 0, \forall f \in L_a^2(\mathbb{B}_n)$$

we have $\xi_k \perp [p]$ and hence $\{\xi_k\} \subseteq \mathcal{Q}_p$. Moreover, $Q_f^* \xi_k = M_f^* \xi_k = \overline{f(\xi_k)} \xi_k$ for $k \in \mathbb{N}, f \in \mathbb{C}[z_1, \cdots, z_n]$.

Now we claim that such $w_0 = (w_1^0, \cdots, w_n^0) \in X_p$. Otherwise, the n-tuple of operators $(Q_{z_1 - w_1^0}, \cdots, Q_{z_n - w_n^0})$ is Fredholm and hence the range of $H = \sum_{i=1}^n Q_{z_i - w_i^0} Q_{z_i - w_i^0}^*$ has finite codimension in $\mathcal{Q}_p$. Thus there exists a finite rank projection E and $\varepsilon > 0$ such that $H + E > \varepsilon I_{\mathcal{Q}_p}$. However, a direct computation shows that

$$\langle (H + E)\xi_k, \, \xi_k \rangle = \sum_{i=1}^n |w_i^k - w_i^0|^2 + \langle E\, \xi_k, \, \xi_k \rangle \to 0,$$

since E is a finite rank operator and $\xi_k \to 0$ weakly. This leads to a contradiction. Therefore, we have $w_0 \in X_p$, completing the proof of the proposition. □

In many cases, the two sets are equal and thus X_p is characterized completely.

Recall that $f \in \mathbb{C}[z_1, \cdots, z_n]$ is said to be quasi-homogeneous if there exists $k_1, \cdots, k_n \in \mathbb{N}$ and a homogeneous polynomial $g \in \mathbb{C}[z_1, \cdots, z_n]$ such that $f(z_1, \cdots, z_n) = g(z_1^{k_1}, \cdots, z_n^{k_n})$ for $(z_1, \cdots, z_n) \in \mathbb{C}^n$.

Corollary 4.7. *For a quasi-homogeneous polynomial $p \in \mathbb{C}[z_1, \cdots, z_n]$, we have $X_p = Z(p) \cap \partial\mathbb{B}_n$.*

Proof. Suppose that $p(z_1, \cdots, z_n) = g(z_1^{k_1}, \cdots, z_n^{k_n})$ for some homogeneous polynomial g and $k_1, \cdots, k_n \in \mathbb{N}$. For any $z_0 = \{z_1^0, \cdots, z_n^0\} \in Z(p) \cap \partial\mathbb{B}_n$, we have $g((z_1^0)^{k_1}, \cdots, (z_n^0)^{k_n}) = 0$, which implies that $g(r(z_1^0)^{k_1}, \cdots, r(z_n^0)^{k_n}) = 0$ for $0 < r < 1$ since g is homogeneous. This means that $p(z^r) = 0$ for $z^r = (r^{\frac{1}{k_1}} z_1^0, \cdots, r^{\frac{1}{k_n}} z_n^0)$, and $z^r \to z^0$, which completes the proof. □

Since $p \in \mathbb{C}[z_1, \cdots, z_n]$ defines the extension of $\mathcal{K}(\mathcal{Q}_p)$ by $C(X_p)$, we have $[p] \in K_1(X_p)$, the odd K-homology group of the compact metrizable space X_p [6]. A basic question is to determine which element one has. In [10] it was conjectured that $[p]$ is the fundamental class of X_p determined by the almost complex structure of $X_p \subset \partial\mathbb{B}_n$. (In [10] the multiplicity of p was not taken into account. For example, one sees that $X_{p^2} = X_p$ for $p \in \mathbb{C}[z_1, \cdots, z_n]$ but $[p^2] = 2[p] \in K_1(X_p)$.) In [17], the element $[p]$ is calculated for the case p is homogeneous and $n = 2$ and in this case one can show that $[p]$ equals the fundamental class. In this case X_p consists of the union of a finite number of circles. Hence $[p] \in K_1(X_p)$ is determined by the index of an appropriate operator for each circle with the property that the fundamental class is determined by the "winding number" of the polynomial on these circles. The basic technique in [17] is to first factor $p(z_1, z_2)$ and reduce the calculation to that of a single factor.

The proposition raises a number of questions which we now discuss briefly.

First, is it always the case that $X_p = clos(Z(p) \cap \mathbb{B}_n) \cap \partial\mathbb{B}_n$? This question is closely related to the question of whether $p \in \mathbb{C}[z_1, \cdots, z_n]$ with $Z(p) \cap \mathbb{B}_n = \phi$ is cyclic which

was answered in the affirmative in [8]. What would be needed to solve the question here would be a technique which allows one to handle the case in which some points in $Z(p) \cap \partial\mathbb{B}_n$ are "tangent" to $\partial\mathbb{B}_n$ but others are not.

Moreover, we observe that the above proposition carries over to many more general submodules of $L^2_a(\mathbb{B}_n)$. For example, for $\phi_1, \cdots, \phi_k \in A(\mathbb{B}_n)$, the ball algebra of functions continuous on $clos(\mathbb{B}_n)$ and holomorphic on $\mathbb{B}_n$, one can see that

$$clos(Z(\phi_1, \cdots, \phi_k) \cap \mathbb{B}_n) \cap \partial\mathbb{B}_n \subseteq X_{[\phi_1, \cdots, \phi_k]} \subseteq Z(p) \cap \partial\mathbb{B}_n,$$

where $[\phi_1, \cdots, \phi_k]$ denotes the submodule of $L^2_a(\mathbb{B}_n)$ generated by $\phi_1, \cdots, \phi_k$ and $Z(\phi_1, \cdots, \phi_k)$ is the subset of $clos\mathbb{B}_n$ of common zeros of $\phi_1, \cdots, \phi_k$. Similarly, the question whether the maximal ideal space $X_{[\phi_1, \cdots, \phi_k]} = clos(Z(\phi_1, \cdots, \phi_k) \cap \mathbb{B}_n) \cap \partial\mathbb{B}_n$ is related to the question of whether $Z(\phi_1, \cdots, \phi_k) \cap \mathbb{B}_n = \phi$ implies $[\phi_1, \cdots, \phi_k] = L^2_a(\mathbb{B}_n)$, which is still open for the dimension $n > 2$. The above argument also extends to other reproducing kernel Hilbert modules such as the Hardy and Drury-Arveson spaces.

Second, the conjecture of Arveson concerns the closure of homogeneous polynomial ideals in the Drury-Arveson space. One can show in the case of homogeneous ideals, essential normality of the closure in the Hardy, Bergman and Drury-Arveson spaces are all equivalent. But this argument doesn't work for the case of ideals generated by arbitrary $p \in \mathbb{C}[z_1, \cdots, z_n]$. It seems likely that the argument in this paper can be generalized to obtain the same result for the Hardy and the Drury-Arveson spaces. However, while we believe that both results hold, perhaps techniques from [9, 7] may be needed to complete the proofs.

Thirdly, in [10] the first author offered a refined conjecture for the closure of homogeneous polynomial ideals in the Drury-Arveson space. Arveson conjectured that the commutators and cross-commutators for the operators Q_f in $\mathscr{Q}_p$ were in the Schatten $p-$class for $p > n$ which we have established in this paper for the case of principal polynomial ideals. However, in [10], it was conjectured that this result on the commutators actually holds for $p > \dim Z(p)$. Although it is not clear if one can modify the proof herein to obtain this result, the question makes sense.

Finally, it is natural to ask if the result in this paper extends to all ideals in $\mathbb{C}[z_1, \cdots, z_n]$ or even to all ideals in $A(\mathbb{B}_n)$. One approach to this problem was discussed in [12]. A question, seemingly beyond current techniques, is whether a submodule of $L^2_a(\mathbb{B}_n)$ is essentially normal if and only if it is finitely generated. However, for the case $n = 1$, the equivalence holds with one direction following from the Berger-Shaw Theorem [5] and the other from the result in [1].

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